# Degeneration mod torsion of the conjugate spectral sequence

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April 17, 2020[*]

## 1 Review of the de Rham-Witt complex

Setup: $X$ is a smooth proper variety over a perfect field $k$ of characteristic $p$. Let $W = W(k)$, $\sigma : W \to W$ the Witt vector Frobenius, and $K = \operatorname{Frac} W$. The de Rham-Witt complex of $X/k$, first constructed by Illusie in 1979, is designed to lift the de Rham complex $\Omega^*_{X/k}$ to characteristic 0, and thereby to compute crystalline cohomology. It is defined as the initial object in a rather complicated category. Rather than giving its full definition, I will just recall what kinds of structure it has, and some of the key conditions we impose. It contains the data:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow R & & \downarrow R & & \\
W_2\mathcal{O}_X & \xrightarrow{d} & W_2\Omega^1_X & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \\
W_1\mathcal{O}_X & \xrightarrow{d} & W_1\Omega^1_X & \xrightarrow{d} & \cdots
\end{array}$$

Here each $W_n\Omega^i_X$ is a sheaf of $W_n\mathcal{O}_X$-modules, with $W_nk$-linear differentials and vertical quotient maps. (The bottom row is just the de Rham complex of $X$, and the leftmost column is the sheaf of Witt vectors of $\mathcal{O}_X$.) Additionally, each row has a multiplication map making it a cdga. Finally, each column has maps $F$ going down and $V$ going up, satisfying the following relations:

(a) $FV = VF = p$,

(b) $dF = pFd$, $Vd = pdV$, $FdV = d$,

(c) $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$,

[*]Notes for a virtual talk in Berkeley's student arithmetic geometry seminar. Main references: Illusie-Raynaud, "Les suites spectrales associées au complexe de de Rham-Witt", and Illusie, "Complexe de de Rham-Witt et cohomologie cristalline".

and various others.

The complex $W\Omega_X^*$ is defined as $\lim_{\leftarrow} W_n\Omega_X^*$. The $F, V$, and $d$ operators and the multiplication map pass to the inverse limit, and they have the same relations as above. Given $W\Omega_X^*$ with all of these operators, we can recover $W_n\Omega_X^*$ as its quotient by the images of $V^n$ and $dV^n$. In practice, we pass between $W\Omega_X^*$ and $(W_n\Omega_X^*)_n$ more or less freely, but one must be somewhat cautious about what operations do and don't commute with the limit.

Remark: Under our smoothness hypotheses, $W\Omega_X^*$ turns out to be $p$-torsion-free. Then each of the relations in (b) above is equivalent to saying that the $\sigma$-semilinear map $\varphi$ defined by $p^iF$ on $W\Omega^i$ commutes with $d$. This is useful because it means the operator $\varphi$ will pass to everything in the next section, including crystalline cohomology, and all of the maps that come up will be compatible with $\varphi$.

## 2 The slope and conjugate spectral sequences

The most fundamental fact about the de Rham-Witt complex is as follows:

Theorem: The (hyper)cohomology of the de Rham-Witt complex computes crystalline cohomology. More precisely, we have isomorphisms

$$H^*_{\text{cris}}(X/W_n) \cong H^*(W_n\Omega_X^*) := R^*\Gamma(W_n\Omega_X^*)$$
$$H^*_{\text{cris}}(X/W) \cong H^*(W\Omega_X^*) := R^*\Gamma(W\Omega_X^*).$$

Given a complex of sheaves $K^*$ equipped with a filtration, there is a spectral sequence allowing us to compute its cohomology in terms of the cohomology of the associated graded objects. There are two natural choices of filtration here, and both give interesting spectral sequences. (I'll discuss the spectral sequences for $W_n\Omega_X^*$; the corresponding statements for $W\Omega_X^*$ follow if we are careful about $R^i\lim$'s.)

The *slope spectral sequence* comes from the *stupid filtration* $\sigma_{\geq i}W_n\Omega_X^*$,

$$\begin{array}{ccccccccccc}
\sigma_{\geq i}W_n\Omega_X^* & & \cdots & \longrightarrow & 0 & \longrightarrow & W_n\Omega_X^i & \longrightarrow & W_n\Omega_X^{i+1} & \longrightarrow & \cdots \\
\big\downarrow & & & & \big\downarrow & & \big\| & & \big\| & & \\
W_n\Omega_X^* & & \cdots & \longrightarrow & W_n\Omega_X^{i-1} & \longrightarrow & W_n\Omega_X^i & \longrightarrow & W_n\Omega_X^{i+1} & \longrightarrow & \cdots
\end{array}$$

with graded pieces $\operatorname{gr}^i W_n\Omega_X^* = W_n\Omega_X^i[-i]$.

It has the form:

$${}_n'E_1^{i,j} = H^j(W_n\Omega_X^i) \implies H^{i+j}(W_n\Omega_X^*) = H^{i+j}_{\text{cris}}(X/W_n) \text{ or}$$
$${}'E_1^{i,j} = H^j(W\Omega_X^i) \implies H^{i+j}(W\Omega_X^*) = H^{i+j}_{\text{cris}}(X/W)$$

(*Notation*: we will always use $'E$ to refer to the first spectral sequence and $''E$ for the second. The left subscript $n$ indicates that we are working over $W_n$ instead of $W$.)

The *conjugate spectral sequence* comes from the *canonical filtration* $\tau_{\leq i} W_n\Omega^*_X$,

$$
\begin{array}{ccccccccccc}
\tau_{\leq i} W_n\Omega^*_X & & \cdots & \longrightarrow & W_n\Omega^{i-1}_X & \longrightarrow & \ker(d^i) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\big\downarrow & & & & \big\| & & \big\downarrow & & \big\downarrow & & \\
W_n\Omega^*_X & & \cdots & \longrightarrow & W_n\Omega^{i-1}_X & \longrightarrow & W_n\Omega^i_X & \longrightarrow & W_n\Omega^{i+1}_X & \longrightarrow & \cdots
\end{array}
$$

with graded pieces

$$
\begin{aligned}
\operatorname{gr}^i W_n\Omega^*_X &= (W_n\Omega^{i-1}_X / \ker(d^{i-1}) \overset{d^{i-1}}{\hookrightarrow} \ker(d^i)) \\
&\overset{\text{q.i.}}{\simeq} \mathscr{H}^i(W_n\Omega^*_X)[-i]
\end{aligned}
$$

Here the $''E_1$ page is not canonical, but the $''E_2$ page is:

$$
\begin{aligned}
{}''_nE_2^{ij} = H^i(X, \mathscr{H}^j(W_n\Omega^*_X)) &\implies H^{i+j}(W_n\Omega^*_X) = H^{i+j}_{\text{cris}}(X/W_n), \text{ or} \\
{}''E_2^{ij} = H^i(X, \mathscr{H}^j(W\Omega^*_X)) &\implies H^{i+j}(W\Omega^*_X) = H^{i+j}_{\text{cris}}(X/W),
\end{aligned}
$$

where $\mathscr{H}^j$ denotes the cohomology sheaves—literally cocycles mod coboundaries.

To make this concrete, suppose we have a sufficiently nice (i.e. Cartan-Eilenberg) resolution $I^{**}$ of the complex $W_n\Omega^i_X$. This is a double complex of sheaves of $W_n$-modules of the form

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{01} & \longrightarrow & I^{11} & \longrightarrow & I^{21} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{00} & \longrightarrow & I^{10} & \longrightarrow & I^{20} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W_n\Omega^0_X & \longrightarrow & W_n\Omega^1_X & \longrightarrow & W_n\Omega^2_X & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where each $I^{ij}$ is injective and each column is a resolution of $W_n\Omega^i_X$.

Then $W_n\Omega^*_X$ is quasi-isomorphic to the total complex $\operatorname{Tot}(I^{**})$. In particular, we can compute its cohomology as the cohomology of $\operatorname{Tot}(\Gamma(I^{**}))$. This in turn can be computed by running a spectral sequence whose $E_0$ page is the following (non-canonical) double complex:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^{02}) & \longrightarrow & \Gamma(I^{12}) & \longrightarrow & \Gamma(I^{22}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^{01}) & \longrightarrow & \Gamma(I^{11}) & \longrightarrow & \Gamma(I^{21}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^{00}) & \longrightarrow & \Gamma(I^{10}) & \longrightarrow & \Gamma(I^{20}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

But given a double complex, there are two different associated spectral sequences. Starting with vertical maps leads to the slope spectral sequence, and starting with horizontal maps leads to the conjugate spectral sequence.

Another interpretation of the conjugate spectral sequence:

I lied earlier: the most fundamental fact about about $W\Omega^*_X$ is *not* that its hypercohomology computes crystalline cohomology. Rather, the global sections functor $\Gamma : (X/W)_{\text{cris}} \to \text{Sh}(*)$ factors through $\text{Sh}(X_{\text{Zar}})$:

$$(X/W)_{\text{cris}} \xrightarrow{u_*} \text{Sh}(X_{\text{Zar}}) \xrightarrow{\Gamma} \text{Sh}(*)$$

and therefore the cohomology functor $R\Gamma : D((X/W)_{\text{cris}}) \to D(\text{Sh}(*))$ factors through $D(\text{Sh}(X_{\text{Zar}}))$:

$$D((X/W)_{\text{cris}}) \xrightarrow{Ru_*} D(\text{Sh}(X_{\text{Zar}})) \xrightarrow{R\Gamma} D(\text{Sh}(*)).$$

Crystalline cohomology is defined as $R\Gamma(\mathcal{O}_{X/W})$. The most fundamental fact about the de Rham-Witt complex is that it's a representative of the derived Zariski sheaf $Ru_*(\mathcal{O}_{X/W})$ as an honest complex of sheaves.

It follows from this that $R\Gamma(W\Omega^*_X)$ equals crystalline cohomology, as $R\Gamma \circ Ru_* = R\Gamma$. The conjugate spectral sequence appears here as the Leray spectral sequence for the composition of two derived functors. In particular, this shows that the conjugate spectral sequence is interesting to study even if a priori we are only interested in crystalline cohomology and not in the de Rham-Witt complex.

Let's compare our two spectral sequences (over $W$):

$$
\begin{aligned}
{}'E_1^{i,j} &= H^j(W\Omega^i_X) \implies H^{i+j}_{\text{cris}}(X/W), \\
{}''E_2^{i,j} &= H^i(X, \mathscr{H}^j(W\Omega^*_X)) \implies H^{i+j}_{\text{cris}}(X/W)
\end{aligned}
$$

Note that there are three differences:

- The first starts at $E_1$ and the second at $E_2$.
- The roles of $i$ and $j$ get switched (because of starting with horizontal vs. vertical maps).
- The first involves the sheaf cohomology of $W\Omega^i_X$ itself, whereas the second involves the sheaf cohomology of the cohomology sheaves of the complex $W\Omega^*_X$.

# 3 Example

For a typical example of what the two spectral sequences look like, let $X$ be a supersingular abelian surface. Then the $'E_1$ page of the slope spectral sequence looks like:

$$\begin{array}{ccc}
k[[x]] \hookrightarrow & k[[x]] \oplus W^{\oplus 4} & W \\
 & & \\
W^{\oplus 4} & W^{\oplus 6} & 0 \\
 & & \\
W & 0 & 0
\end{array}$$

The $''E_2$ page of the spectral sequence looks (I think) like:

$$\begin{array}{cccc}
0 & k[[x]] & & W \\
0 & W^{\oplus 6} & \nwarrow \simeq & W^{\oplus 4} \\
W & W^{\oplus 4} & & k[[x]]
\end{array}$$

In both cases, all maps are zero except for the indicated maps on torsion, and the spectral sequences degenerate on the following page with no torsion.

# 4 Recap of Illusie's results

Proposition (Illusie): For all $i, j$, and $n$, the $W_n$-module $H^j(W_n\Omega^i_X)$ has finite length.

Remark: This result is needed to prove that $H^j(W\Omega^i_X) = \lim_{\leftarrow n} H^j(W_n\Omega^i_X)$, so that the slope spectral sequence over $W$ is the inverse limit of the ones over $W_n$. Note that in our example, some $H^j(W_n\Omega^i_X)$ have infinitely much $p$-torsion, but only finitely much of this appears over any given $W_n$.

Before stating Illusie's main result, let me briefly explain the "slope" terminology. The undivided Frobenius $\varphi$ mentioned earlier induces operators $\varphi$ on each $H^j(W\Omega^i)$, and therefore on $H^j(W\Omega^i)/\operatorname{tors}$. These are Frobenius-semilinear maps on finite free $W$-modules. Any such object has a collection of *slopes*, which are the semilinear analogues of $p$-adic valuations of eigenvalues. Since the divided Frobenius $F$ satisfies $FV = p$, with $V$ topologically nilpotent, it must have all its slopes in $[0, 1)$. It follows that $\varphi = p^i F$ on $H^j(W\Omega^i_X)$ has slopes in $[i, i+1)$. In fact this discussion implies the theorem:

Theorem (Illusie): The slope spectral sequence degenerates at ${}'E_1$ mod torsion (i.e. after applying $\otimes_W K$), and the $i$-th graded piece of the induced filtration is the part of $H^*_{\text{cris}}(X/W) \otimes_W K$ with slope in $[i, i+1)$.

Proof: All ${}'E_1^{i,j}$ have $\varphi$ operators with slopes in $[i, i+1)$, and all differentials respect $\varphi$. It follows that the ${}'E_n^{i,j}$ for $n \geq 1$ inherit $\varphi$, also with slopes in $[i, i+1)$, and also commuting with differentials. But the differentials on page ${}'E_1$ and beyond go between modules with no slopes in common, so they're 0 mod torsion.

# 5 Results of Illusie-Raynaud

Lemma: The ${}''_nE_2$ page of the conjugate spectral sequence is valued in finite-length $W_n$-modules.

Proof: For each $n$, we have a so-called *higher Cartier isomorphism*

$$C^{-n} : W_n\Omega^i_X \xrightarrow{\sim} \mathscr{H}^i(W_n\Omega^*_X),$$

which is a $\sigma^n$-semilinear isomorphism of sheaves of $W_n$-modules on $X$. (In the case $n = 1$, this is the usual Cartier isomorphism.) Taking cohomology on both sides gives us $\sigma^n$-semilinear isomorphisms

$${}'_nE_1^{ij} = H^j(X, W_n\Omega^i_X) \cong H^j(X, \mathscr{H}^i(W_n\Omega^*_X)) = {}''_nE_2^{ji}.$$

Since the left side has finite length, the right side does too.

Warning: These isomorphisms are *not* compatible as $n$ varies, so we cannot get an isomorphism of objects over $W$ by passing to the limit.

Main theorem: The conjugate spectral sequence degenerates at ${}''E_2$ mod torsion (i.e. after applying $\otimes_W K$), and the $j$-th graded piece of the induced filtration is the part of $H^*(X/W) \otimes_W K$ with slope in $(j-1, j]$.

Sketch of proof: Recall that in Illusie's proof of degeneration, the key idea was as follows. The object ${}'E_1^{ij} = H^j(W\Omega^i_X)$ comes with $F$ and $V$ operators, such that $\varphi = p^i F$ is compatible with the maps in the spectral sequence and has slopes in $[i, i+1)$. Since these intervals are disjoint for different $i$, it followed that all maps in the spectral sequence (mod torsion) vanish.

We want to imitate this for ${}''E_2^{ij} = H^i(\mathscr{H}^j(W\Omega^*_X))$. The problem is that neither $F$ nor $V$ induces a well-defined operator on $\mathscr{H}^j(W\Omega^*_X)$: $F$ preserves the cocycles $ZW\Omega^i_X$ but not the coboundaries $BW\Omega^i_X$, and $V$ preserves coboundaries but not cocycles. Instead we define $F' = pF : W\Omega^i \to W\Omega^i$ and $V' = F^{-1}|_{ZW\Omega^i_X}$. These maps preserve both cocycles and coboundaries, so they induce maps on $\mathscr{H}^j(W\Omega^*_X)$ and thus $H^i(\mathscr{H}^j(W\Omega^*_X))$. These have the right semilinearity properties, and they satisfy $F'V' = V'F' = p$.

The operator $F' = pF$ is topologically nilpotent on $W\Omega^i$, since $p$ is. It follows that $F'$ is topologically nilpotent (albeit no longer divisible by $p$) as an operator on $\mathscr{H}^j(W\Omega_X^*)$, and also on $H^i(\mathscr{H}^j(W\Omega_X^*))/\operatorname{tors}$. Then the slopes of $F'$ are in $(0, 1]$, so the slopes of $\varphi = p^j F = p^{j-1}F'$ are in $(j-1, j]$. From here the proof concludes as in Illusie.

So what's the catch? This is a 140-page paper, right?

Showing that $F'$ and $V'$ give well-defined maps on $\mathscr{H}^j(W\Omega_X^*)$ takes some work, but not that much. But the main issue is that in order to talk about slopes, we need to know that $H^i(\mathscr{H}^j(W\Omega_X^*))/\operatorname{tors}$ is a finite free $W$-module. Illusie-Raynaud states this as another part of their main theorem, along with saying that $H^i(\mathscr{H}^j(W\Omega_X^*))$ has bounded $p$-power torsion. Proving it requires a precise understanding of what kind of object $H^i(\mathscr{H}^j(W\Omega_X^*))$ is.

# 6 Graded $R$-modules

Let $R$ denote the noncommutative graded $W$-algebra generated by elements $F$ and $V$ in degree 0 and $d$ in degree 1, subject to all the usual relations:

- $FV = VF = p$, $d^2 = 0$,
- $dF = pFd$, $Vd = pdV$, $FdV = d$,
- $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$.

This ring is concentrated in degrees 0 and 1. It is a free $W$-module with basis:

$$\{F^m, V^n, F^m d, dV^n : m \geq 0, n > 0\}.$$

Any complex of $W$-modules with suitable $F$ and $V$ operators is then a (graded left) $R$-module. Given such a module $M^*$, we define

$$W_n M^i = M^i/(V^n M^i, dV^n M^{i-1}).$$

Illusie-Raynaud then proves a wonderful structure theorem for sufficiently "nice" $R$-modules:

Proposition: Suppose $M^*$ is a graded left $R$-module, concentrated in finitely many degrees, such that $M^* = \lim_{\leftarrow n} W_n M^*$ and each $W_n M^i$ is a finite-length $W$-module. Then $M^*$ has a finite filtration with quotients of the following types:

1. Concentrated in one degree:
   (a) finite-length torsion $W$-modules,
   (b) finite free $W$-modules,
   (c) $k_\sigma[[V]]$, with $F = 0$,
2. "dominoes," denoted $U_i[-n]$, concentrated in degrees $n$ and $n+1$.

In particular, each $''E_2^{ij} = H^i(\mathscr{H}^j(W\Omega_X^*))$ satisfies these hypotheses, so one can use the proposition to prove finiteness properties about it.

# 7 Applications

Theorem (Rudakov-Shafarevich) A K3 surface $X$ over an arbitrary field $k$ has no global vector fields. This is easy in characteristic 0. In the characteristic-$p$ case, they use Illusie-Raynaud's theory of dominoes to study various differentials in the Hodge-de Rham, slope, and conjugate spectral sequences, and eventually show that $H^0(X, T_X) \cong H^0(X, \Omega^1_X) = 0$.

Ekedahl's thesis uses some further study of the category of $R$-modules to show how one can prove Künneth and duality formulas for crystalline cohomology using the de Rham-Witt complex.

Katz ("Crystalline cohomology, Dieudonné modules, and Jacobi sums", 1981) gives a formula for Gauss sums using the degeneration of the conjugate spectral sequence for Artin-Schreier covers of $\mathbb{P}^1$.